

Doubling inequalities for the Lamé system with rough coefficients

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Abstract

In this paper we study the local behavior of a solution to the Lamé system when the Lamé coefficients λ and μ satisfy that μ is Lipschitz and λ is essentially bounded in dimension $n \geq 2$. One of the main results is the *local* doubling inequality for the solution of the Lamé system. This is a quantitative estimate of the strong unique continuation property. Our proof relies on Carleman estimates with carefully chosen weights. Furthermore, we also prove the *global* doubling inequality, which is useful in some inverse problems.

1 Introduction

Let Ω be an open connected subset of \mathbb{R}^n with $n \geq 2$. Without loss of generality, we assume $0 \in \Omega$. Let $\mu(x) \in C^{0,1}(\Omega)$ and $\lambda(x), \rho(x) \in L^\infty(\Omega)$ satisfy

$$\begin{cases} \mu(x) \geq \delta_0, & \lambda(x) + 2\mu(x) \geq \delta_0 \quad \forall x \in \Omega, \\ \|\mu\|_{C^{0,1}(\Omega)} + \|\lambda\|_{L^\infty(\Omega)} \leq M_0, & \|\rho\|_{L^\infty(\Omega)} \leq M_0 \end{cases} \quad (1.1)$$

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with positive constants δ_0, M_0 , where we define

$$\|f\|_{C^{0,1}(\Omega)} = \|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{L^\infty(\Omega)}.$$

The isotropic elasticity system is given by

$$\operatorname{div}(\mu(\nabla u + (\nabla u)^t)) + \nabla(\lambda \operatorname{div} u) + \rho u = 0 \quad \text{in } \Omega, \quad (1.2)$$

where $u = (u_1, u_2, \dots, u_n)^t$ is the displacement vector and $(\nabla u)_{jk} = \partial_k u_j$ for $j, k = 1, 2, \dots, n$. If $\rho = 0$, (1.2) represents the displacement equation of equilibrium.

Under the assumptions (1.1), the qualitative strong unique continuation property for (1.2) were recently proved by Nakamura, Uhlmann and the second and third authors [12], i.e., if $u \in H^1(\Omega)$ solves (1.2) and satisfies that for any $N \in \mathbb{N}$, there exists a constant C_N such that

$$\int_{B_r} |u|^2 \leq C_N r^N \quad \forall r \text{ sufficiently small},$$

then $u \equiv 0$ in Ω . In fact, in [12], we derived a quantitative estimate on the vanishing order of any nontrivial solution to (1.2). The derivation relies on the *optimal* three-ball inequalities (see [12] for details).

Another quantitative estimate of the strong unique continuation property is the doubling inequality. When $\lambda, \mu \in C^{1,1}$ and $\rho = 0$, doubling inequalities for (1.2) in the form

$$\int_{B_{2r}} |u|^2 + |\operatorname{div} u|^2 \leq K \int_{B_r} |u|^2 + |\operatorname{div} u|^2.$$

were derived in [1] based on the frequency function method developed in [5] and [6]. To apply quantitative estimates of the strong unique continuation property to certain inverse problems for the elasticity, it is desirable to derive a doubling inequality containing $|u|^2$ only [2], i.e.,

$$\int_{B_{2r}} |u|^2 \leq K \int_{B_r} |u|^2. \quad (1.3)$$

Indeed, (1.3) for the Lamé system with $C^{1,1}$ coefficients was proved in [2]. However, as mentioned in [3], the proof given there contains a gap.

In [3], the authors proved doubling inequalities of the form (1.3) when $\lambda, \mu \in C^{2,1}$ (also $\rho = 0$). Moreover, these inequalities depend on global

properties of the solution. A key observation in [3] is that for $\lambda, \mu \in C^{2,1}$, the Lamé system can be transformed into a fourth order system for u having Δ^2 as the leading part and essentially bounded coefficients in the lower orders. For this fourth order system, three-sphere inequalities and local doubling inequalities were derived in [9]. Using these inequalities, global doubling inequalities for (1.2) were then obtained.

The aim of this paper is to establish doubling inequalities of the form (1.3) for (1.2) when $\mu \in C^{0,1}$ and $\lambda, \rho \in L^\infty$. Our result provides a positive answer to the open problem posed in [3] about the doubling inequality for (1.2) with less regular coefficients. The ideas of our proof originate from our series papers on proving quantitative uniqueness for elliptic equations or systems by the method of Carleman estimates [10], [11], and [12]. In particular, we will use the reduced system derived in [12] (see Section 2 below).

We now state main results of the paper. Their proofs will be given in the subsequent sections. Assume that there exists $0 < \tilde{R}_0 \leq 1$ such that $B_{\tilde{R}_0} \subset \Omega$. Hereafter B_r denotes an open ball of radius $r > 0$ centered at the origin.

Theorem 1.1 *There exists $C > 0$ depending on n , M_0 and δ_0 so that the following is true. If $R > 0$ with $3R \leq \tilde{R}_0$, $u \in H_{loc}^1(B_R)$ is a nonzero solution to (1.2) and*

$$m = -\ln \left(\frac{\|u\|_{L^2(B_R \setminus B_{R/2})}}{\|u\|_{L^2(B_{2R} \setminus B_R)}} \right)$$

then

$$\|u\|_{L^2(B_r)} \geq C(r/R)^{Cm} \|u\|_{L^2(B_{2R} \setminus B_R)} \quad \text{for all } r \leq R. \quad (1.4)$$

Theorem 1.2 *There exists a positive constant \tilde{C} depending only on n , M_0 and δ_0 such that the following is true. If $u \in H_{loc}^1(B_R)$ is a nonzero solution to (1.2) then*

$$\|u\|_{L^2(B_{2r}(x_0))} \leq \tilde{C} e^{\tilde{C}m} \|u\|_{L^2(B_r(x_0))}, \quad (1.5)$$

whenever $B_{2r}(x_0) \subset B_{R/2}$. Here m is the constant of Theorem 1.1.

Theorem 1.1 and 1.2 will be proved together. The estimate (1.5) is called local doubling inequalities. Global doubling inequalities in which constants depend on the global property of solution will be proved in Section 4.

2 Reduced system

We now recall the reduced system derived in (1.2). This is a crucial step in our approach. Let us write (1.2) into a non-divergence form:

$$\mu\Delta u + \nabla((\lambda + \mu) \operatorname{div} u) + (\nabla u + (\nabla u)^t)\nabla\mu - \operatorname{div} u \nabla\mu + \rho u = 0. \quad (2.1)$$

Dividing (2.1) by μ yields

$$\begin{aligned} & \Delta u + \frac{1}{\mu}\nabla((\lambda + \mu) \operatorname{div} u) + (\nabla u + (\nabla u)^t)\frac{\nabla\mu}{\mu} - \operatorname{div} u \frac{\nabla\mu}{\mu} + \frac{\rho}{\mu}u \\ = & \Delta u + \nabla(a(x)p) + G \\ = & 0, \end{aligned} \quad (2.2)$$

where

$$a(x) = \frac{\lambda + \mu}{\lambda + 2\mu} \in L^\infty(\Omega), \quad p = \frac{\lambda + 2\mu}{\mu} \operatorname{div} u$$

and

$$G = (\nabla u + (\nabla u)^t)\frac{\nabla\mu}{\mu} - \operatorname{div} u \left(\frac{\nabla\mu}{\mu} + (\lambda + \mu)\nabla\left(\frac{1}{\mu}\right)\right) + \frac{\rho}{\mu}u.$$

Taking the divergence on (2.2) gives

$$\Delta p + \operatorname{div} G = 0. \quad (2.3)$$

Our reduced system now consists of (2.2) and (2.3). It follows easily from (2.3) that if $u \in H_{loc}^1(\Omega)$, then $p \in H_{loc}^1(\Omega)$.

$$\begin{cases} \Delta u + \nabla(a(x)p) + G(x, u) = 0, \\ \Delta p + \operatorname{div} G(x, u) = 0. \end{cases} \quad (2.4)$$

Note that system (2.4) is not decoupled. We will use (2.4) to prove our theorems.

3 Proofs of Theorem 1.1 and 1.2

This section is devoted to the proofs of Theorem 1.1 and 1.2. The proofs rely on a suitable Carleman estimate proved in [8]. To state the estimate, we consider the equation

$$\Delta u + \nabla f = g \quad \text{in } \mathbb{R}^n. \quad (3.1)$$

We consider $t > 0$. Given $\tau \gg 1$, let $h(t)$ be a convex function satisfying

$$\begin{cases} h' \sim \tau, \text{ i.e., } \exists C > 1, C^{-1}\tau \leq h' \leq C\tau, \\ \text{dist}(2h', \mathbb{Z}) + h'' \gtrsim 1. \end{cases} \quad (3.2)$$

Here and in the sequel, the notation $X \lesssim Y$ or $X \gtrsim Y$ means that $X \leq CY$ or $X \geq CY$ with some constant C depending only on n , M_0 and δ_0 . We further assume that h satisfies that for any $C > 0$ there exists $R_0 > 0$ such that

$$C|x|\tau \leq (1 + h''(-\ln|x|)) \quad (3.3)$$

for all τ and $|x| \leq R_0$. Given $R > 0$ $h(-\ln(R_0x/R))$ satisfies 3.3 for $|x| \leq R$.

For our purpose, in addition to (3.2), we also require $h - t - \frac{1}{2}\ln(1 + h'')$ to satisfy (3.2). The existence of such weight function h can be found in [8, Section 6]. We will give a more explicit construction of h in appendix.

Theorem 3.1 *Assume that a convex h satisfies (3.2) and is evaluated at $-\ln|x|$. For smooth functions u, f, g satisfying (3.1) and are supported in $B_1(0) \setminus \{0\}$, we have that*

$$\begin{aligned} & \tau \| |x|^{-2}(1 + h'')^{\frac{1}{2}} e^h u \| + \| |x|^{-1}(1 + h'')^{\frac{1}{2}} e^h \nabla u \| \\ & \lesssim \tau \| |x|^{-1} e^h f \| + \| e^h g \|, \end{aligned} \quad (3.4)$$

where $\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^n)}$.

Theorem 3.1 can be proved by adopting arguments of Proposition 4.1 and 4.2 in [8] (see also [4, Proposition 5.1]). It can be also proved by modifying the method in [10]. Here we give a sketch of proof.

Proof. We first observe that the estimate is equivalent to the some estimates for functions on $\mathbb{R}^n \setminus \{0\}$ under appropriate decay conditions of the solutions at 0 and ∞ . This is seen by truncating, and taking an obvious limit.

We begin with an elliptic reduction and consider the equation

$$-\Delta w + K\tau^2|x|^{-2}w = \nabla f$$

with a fixed large positive constant K . The quadratic form

$$\int |\nabla w|^2 dx + K\tau^2 \int |w|^2 |x|^{-2} dx$$

is an inner product and the Riesz representation theorem ensures that there is a unique solution. We claim that

$$\int e^{2h(-\ln|x|)}(|\nabla w|^2 + \tau^2|w|^2/|x|^2)dx \leq C \int e^{2h(-\ln|x|)}|f|^2dx \quad (3.5)$$

for all h satisfying the first condition of (3.2). It suffices to consider bounded functions h and the inequality follows by multiplying by $e^{2h(-\ln|x|)}w$ and integrating by parts. Moreover w decays fast as $x \rightarrow 0$ or $|x| \rightarrow \infty$ which we see by choosing h growing fast and linearly at $\pm\infty$.

We make the ansatz

$$u = v + w$$

where

$$\Delta v = -K\tau^2|x|^{-2}w + g$$

and the full estimate (3.4) follows once we prove the estimate for $f = 0$ and apply it to v . Without loss of generality we assume $f = 0$ in the sequel and prove (3.4).

To prove it in this case, we introduce polar coordinates in $\mathbb{R}^n \setminus \{0\}$ by setting $x = r\omega$, with $r = |x|$, $\omega = (\omega_1, \dots, \omega_n) \in S^{n-1}$. Using the new coordinate $t = -\log r$, we obtain that

$$|x|^{\frac{2+n}{2}} \Delta(|x|^{\frac{2-n}{2}} u) = u_{tt} + \Delta_{S^{n-1}} u - \left(\frac{n-2}{2}\right)^2 u = e^{-\frac{n+2}{2}t} g(e^{-t}\omega).$$

We can diagonalize $-\Delta_{S^{n-1}} + \left(\frac{n-2}{2}\right)^2$. Its spectrum is

$$\left\{\left(\frac{n-2}{2} + k\right)^2 := \sigma_k^2 : k = 0, 1, \dots\right\}$$

and the corresponding eigenspace is spanned by harmonic polynomials. The equation becomes

$$u_{tt}^k - \sigma_k^2 u^k = e^{-\frac{n+2}{2}t} g^k$$

and the estimate (3.4) follows from (including an additional linear term into h without changing the notation)

$$\int e^{2h}(1+h'')(|u_t^k|^2 + (1+k^2)|u^k|^2 + |h'|^2|u^k|^2)dt \leq C \int e^{2h-4t}|g^k|^2 dt$$

Since $\partial_t^2 - \sigma_k^2 = (\partial_t - \sigma_k)(\partial_t + \sigma_k)$, the claim follows once we prove the elliptic estimate

$$\int e^{2h}(|u'|^2 + (\tau^2 + \sigma^2)|u|^2)dt \leq C \int e^{2h}|(\partial_t - \sigma)u|^2 dt \quad (3.6)$$

and the commutator type estimate

$$\int e^{2h}(1 + h'')|u|^2 dt \leq c \int e^{2h}|(\partial_t + \sigma)u|^2 dt \quad (3.7)$$

In the first case we multiply

$$u_t - \sigma u = g$$

by $e^{2h}u$ and integrate. Then

$$\frac{1}{2}(\tau + \sigma)\|e^h u\|^2 \leq \int e^{2h}(h' + \sigma)u^2 dt = - \int e^{2h}ug dt \leq \|e^h g\| \|e^h u\|$$

together with using the equation to bound u_t implies (3.6). For the second estimate we define $v = e^h u$, multiply

$$v' - (h' - \sigma)v = e^h g$$

by $(h' - \sigma)v$ and obtain

$$\|(h' - \sigma)v\|^2 + \frac{1}{2} \int h'' v^2 dt = - \int e^h g(h' - \sigma)v dt.$$

The estimate follows by an application of the Cauchy-Schwarz inequality. \square

Besides the Carleman estimate, we also need an interior estimate (Caccioppoli-type estimate) for the Lamé system (1.2). For fixed $a_3 < a_1 < a_2 < a_4$, there exists a constant C_1 such that

$$\int_{a_1 r < |x| < a_2 r} ||x|^{|\alpha|} D^\alpha u|^2 + ||x|^{|\alpha|+1} D^\alpha p|^2 dx \leq C_1 \int_{a_3 r < |x| < a_4 r} |u|^2 dx, \quad |\alpha| \leq 1 \quad (3.8)$$

for all sufficiently small r . Estimate (3.8) can be found in Lemma 3.1 of [12].

We are now ready to prove Theorem 1.1 and 1.2. Let us define the cut-off function $\chi(x) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$\chi(x) = \begin{cases} 0 & \text{if } |x| \leq r/3, \\ 1 & \text{in } 5r/12 \leq |x| \leq 5\tilde{R}/4, \\ 0 & \text{if } 3\tilde{R}/2 \leq |x|, \end{cases}$$

where \tilde{R} is a small number that will be chosen later and $r \ll \tilde{R}$. Denote $\tilde{u} = \chi u$ and $\tilde{p} = \chi p$. Then it follows from (2.4) that \tilde{u} and \tilde{p} satisfy

$$\Delta \tilde{u} + \nabla(a\tilde{p}) = (\nabla^2 \chi)u + 2\nabla \chi \cdot \nabla u + (\nabla \chi)ap - \chi G := F \quad (3.9)$$

and

$$\Delta \tilde{p} + \operatorname{div}(\chi G) = (\nabla^2 \chi)p + 2\nabla \chi \cdot \nabla p + (\nabla \chi)G := H. \quad (3.10)$$

Applying (3.4) to (3.9) with $u = \tilde{u}$, $f = a\tilde{p}$, $g = F$ yields

$$\begin{aligned} & \tau \| |x|^{-2}(1+h'')^{\frac{1}{2}}e^h \tilde{u} \| + \| |x|^{-1}(1+h'')^{\frac{1}{2}}e^h \nabla \tilde{u} \| \\ & \lesssim \tau \| |x|^{-1}e^h a\tilde{p} \| + \| e^h F \| \\ & \leq C(\tau \| |x|^{-1}e^h \tilde{p} \| + \| e^h F \|), \end{aligned} \quad (3.11)$$

where $C = C(n, M_0, \delta_0)$. Replacing h by $h - t - \frac{1}{2} \ln(1+h'')$ in (3.4) and applying the new estimate to (3.10), we have that

$$\tau \| |x|^{-1}e^h \tilde{p} \| + \| e^h \nabla \tilde{p} \| \lesssim \tau \| (1+h'')^{-\frac{1}{2}}e^h \chi G \| + \| |x|(1+h'')^{-\frac{1}{2}}e^h H \|. \quad (3.12)$$

Now, $K \times (3.12) + (3.11)$ gives

$$\begin{aligned} & \tau \| |x|^{-2}(1+h'')^{\frac{1}{2}}e^h \tilde{u} \| + \| |x|^{-1}(1+h'')^{\frac{1}{2}}e^h \nabla \tilde{u} \| + K\tau \| |x|^{-1}e^h \tilde{p} \| \\ & \leq C(\tau \| |x|^{-1}e^h \tilde{p} \| + \| e^h F \| + K\tau \| (1+h'')^{-\frac{1}{2}}e^h \chi G \| + K \| |x|(1+h'')^{-\frac{1}{2}}e^h H \|). \end{aligned} \quad (3.13)$$

We then choose $K \geq C$ and $\tilde{R} = R(n, M_0, \delta_0)$ satisfying

$$CK\tau(1+h'')^{-\frac{1}{2}} \leq |x|^{-1}(1+h'')^{\frac{1}{2}}$$

for all $|x| \leq 3R/2$ since (3.3) holds. Consequently, we obtain from (3.13) that

$$\begin{aligned} & \tau \| |x|^{-2}(1+h'')^{\frac{1}{2}}e^h u \|_{\{5r/12 \leq |x| \leq 5R/4\}} \leq C \| e^h F \|_{\{r/3 \leq |x| \leq 5r/12\} \cup \{5R/4 \leq |x| \leq 3R/2\}} \\ & + C\tau \| (1+h'')^{-\frac{1}{2}}e^h G \|_{\{r/3 \leq |x| \leq 5r/12\} \cup \{5R/4 \leq |x| \leq 3R/2\}} \\ & + C \| |x|(1+h'')^{-\frac{1}{2}}e^h H \|_{\{r/3 \leq |x| \leq 5r/12\} \cup \{5R/4 \leq |x| \leq 3R/2\}} := RHS. \end{aligned} \quad (3.14)$$

Here and after, we use $\|\cdot\|_A$ to denote the L^2 norm over the region A .

In view of (3.8), we can easily derive that

$$RHS \leq C\tau e^{\tilde{h}(r/3)}(r/3)^{-2}\|u\|_{\{r/4 \leq |x| \leq r/2\}} + C\tau e^{\tilde{h}(5R/4)}(5R/4)^{-2}\|u\|_{\{R/2 \leq |x| \leq 2R\}}, \quad (3.15)$$

where we denote $\tilde{h}(a) = h(-\ln a)$. Now we choose $\tau = \tau_0$ such that

$$Ce^{\tilde{h}(5R/4)}(5R/4)^{-2}\|u\|_{\{R/2 \leq |x| \leq 2R\}} \leq \frac{1}{2}R^{-2}e^{\tilde{h}(R)}\|u\|_{\{2R/3 \leq |x| \leq R\}}. \quad (3.16)$$

More precisely we choose from now on

$$\tau_0 \sim \ln \left(\frac{\|u\|_{\{2R/3 \leq |x| \leq R\}}}{\|u\|_{\{R/2 \leq |x| \leq 2R\}}} \right).$$

so that (3.16) is satisfied. Combining (3.14), (3.15), (3.16) yields

$$\||x|^{-2}e^h u\|_{\{5r/12 \leq |x| \leq 5R/4\}} \leq Ce^{\tilde{h}(r/3)}(r/3)^{-2}\|u\|_{\{r/4 \leq |x| \leq r/2\}}. \quad (3.17)$$

The estimate implies that

$$\|u\|_{\{|x| \leq r\}} \geq Ce^{\tilde{h}(R)}\|u\|_{\{2R/3 \leq |x| \leq R\}}(r/R)^2 e^{-\tilde{h}(r/3)} \geq Cr^m,$$

which establishes Theorem 1.1. Next, adding $e^{h(r/2)}(r/2)^{-2}\|u\|_{\{|x| \leq r/2\}}$ to both sides of (3.17) gives

$$\begin{aligned} e^{\tilde{h}(r)}r^{-2}\|u\|_{\{|x| \leq r\}} &\leq e^{\tilde{h}(r)}r^{-2}\|u\|_{\{|x| \leq r/2\}} + e^{\tilde{h}(r)}r^{-2}\|u\|_{\{r/2 \leq |x| \leq r\}} \\ &\leq Ce^{\tilde{h}(r/3)}(r/3)^{-2}\|u\|_{\{|x| \leq r/2\}}, \end{aligned}$$

which leads to Theorem 1.2.

4 Global doubling inequalities

In the previous section, we have proved local doubling inequalities. Nonetheless, global doubling inequalities are more suitable for inverse problems (for example, see [2]). In this section we derive global doubling inequalities along the lines in [3]. For brevity, we will not give detailed arguments here. We refer to [3] for detailed proofs. To begin, we give the definition of Lipschitz boundary.

Definition 4.1 *We say that the boundary $\partial\Omega$ is of Lipschitz class with constants r_0 and L_0 , if, for any $x_0 \in \partial\Omega$, there exists a rigid transformation of coordinates under which $x_0 = 0$ and*

$$\Omega \cap B_{r_0}(0) = \{x \in B_{r_0}(0) : x_n > \psi(x')\},$$

where $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and ψ is a Lipschitz continuous function on $B_{r_0}(0) \subset \mathbb{R}^{n-1}$ satisfying $\psi(0) = 0$ and

$$\|\psi\|_{C^{0,1}(B_{r_0}(0))} \leq L_0 r_0.$$

Let us denote $\Omega_d = \{x \in \Omega : \text{dist}(x, \partial\Omega) > d\}$. Using three-ball inequalities proved in [11] or [12], one can prove the following theorem (see [2], [3]).

Theorem 4.1 [3, Theorem 3.2] *Let $\partial\Omega$ be of Lipschitz class with constants r_0 , L_0 , and λ , μ satisfy (1.1), $u \in H_{loc}^1(\Omega)$ be a nontrivial solution to (1.2). Then for every $\sigma > 0$ and for every $x \in \Omega_{\frac{4\sigma}{\theta}}$, we have*

$$\int_{B_\sigma(x)} |u|^2 dx \geq C_\sigma \int_\Omega |u|^2 dx,$$

where $0 < \theta < 1$ depends on n , δ_0 , M_0 only and C_σ depends on n , δ_0 , M_0 , r_0 , L_0 , $|\Omega|$, $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$, and σ .

We now ready to state global doubling inequalities. To describe the theorem, we introduce more notations. Instead of the strong ellipticity, we say that Lamé coefficients λ , μ satisfy the strong convexity condition if

$$\mu(x) \geq \tilde{\delta}_0 > 0, \quad 2\mu(x) + n\lambda(x) \geq \tilde{\delta}_0 \quad \forall \quad x \in \Omega. \quad (4.1)$$

It is known that the strong convexity implies the strong ellipticity. Let $\varphi \in L^2(\partial\Omega, \mathbb{R}^n)$ be a vector field satisfying the compatibility condition

$$\int_{\partial\Omega} \varphi \cdot r ds = 0$$

for every infinitesimal rigid displacement r , that is, $r = c + Wx$, where c is a constant vector and W is a skew $n \times n$ matrix. Consider the boundary value problem:

$$\begin{cases} \text{div}(\mu(\nabla u + (\nabla u)^T)) + \nabla(\lambda \text{div} u) = 0 & \text{in } \Omega, \\ (\mu(\nabla u + (\nabla u)^T) + (\lambda \text{div} u) I_n) \nu = \varphi & \text{on } \partial\Omega, \end{cases} \quad (4.2)$$

where I_n is the $n \times n$ identity matrix, ν is the unit outer normal to $\partial\Omega$, and φ satisfies the compatibility condition. In order to ensure the uniqueness of the solution to (4.2), we assume the following normalization conditions:

$$\int_{\Omega} u dx = 0, \quad \int_{\Omega} (\nabla u - (\nabla u)^T) dx = 0. \quad (4.3)$$

Theorem 4.2 [3, Theorem 3.7] *Let $\partial\Omega$ be of Lipschitz class with constants r_0 , L_0 , and λ , μ satisfy (4.1), the second condition of (1.1). If $u \in H^1(\Omega, \mathbb{R}^n)$ is the weak solution to (4.2) satisfying the normalization condition (4.3). Then there exists a constant $0 < \vartheta < 1$, only depending on n , $\tilde{\delta}_0$, M_0 , such that for every $\bar{r} > 0$ and for every $x_0 \in \Omega_{\bar{r}}$, we have*

$$\int_{B_{2r}(x_0)} |u|^2 dx \leq C \int_{B_r(x_0)} |u|^2 dx$$

for every r with $0 < r \leq \frac{\vartheta}{2}\bar{r}$, where C depends on n , $\tilde{\delta}_0$, r_0 , L_0 , $|\Omega|$, \bar{r} , and $\|\varphi\|_{H^{-1/2}(\partial\Omega)} / \|\varphi\|_{H^{-1}(\partial\Omega)}$.

Appendix

In this appendix, we would like to construct a weight function h satisfying the conditions described in Section 3. Let $\tau \in \mathbf{N} + \frac{5}{4} \gg 1$ and define $a = 2 \ln \tau$. We choose

$$h''(t) = \delta \tau e^{-t/2},$$

where $\delta > 0$ is sufficiently small. We then set

$$h'(t) = \tau - 2\delta \tau e^{-t/2}$$

and

$$h(t) = \tau t + 4\delta \tau e^{-t/2}.$$

It is clear that h is convex and h' satisfies the first condition of (3.2). To verify the second condition of (3.2), we observe that $\tau e^{-t/2} \leq 1$ if $t \geq 2 \ln \tau (= a)$ and $\tau e^{-t/2} \geq 1$ if $t \leq 2 \ln \tau$. So, for $t \leq a$, we have $h''(t) \geq \delta \tau e^{-t/2} \geq C_{\delta}(1 + \tau e^{-t/2})$ for some $C_{\delta} > 0$. Next, for $a < t$, we can see that $\tau - 2\delta \leq h'(t) \leq \tau$, then $\text{dist}(2h', \mathbb{Z}) \geq \frac{1}{2} - 4\delta \geq C(1 + \tau e^{-t/2})$ holds for some absolute constant $C > 0$ provided $\delta \leq \frac{1}{16}$.

To check (3.3), as we noted above, if $t \leq 2 \ln \tau$, then

$$1 + h''(-\ln |x|) \geq 1 + \delta\tau\sqrt{|x|} \geq \delta\tau|x|$$

for $|x| < 1$. On the other hand, for $t \geq 2 \ln \tau$, we have

$$1 + h''(-\ln |x|) \geq 1 \geq \tau e^{\ln |x|/2} = \tau\sqrt{|x|} \geq \tau|x|.$$

Finally, let us define $\tilde{h} = h - t - \frac{1}{2} \ln(1 + h'')$, then we have

$$\tilde{h}''(t) = \delta\tau e^{-t/2} - \frac{1}{8}\delta\tau e^{-t/2}(1 + \delta\tau e^{-t/2})^{-1} + \frac{1}{8}\delta^2\tau^2 e^{-t}(1 + \delta\tau e^{-t/2})^{-2}.$$

We choose

$$\tilde{h}'(t) = \tau - 2\delta\tau e^{-t/2} - 1 + \frac{1}{4}\delta\tau e^{-t/2}(1 + \delta\tau e^{-t/2})^{-1}$$

and

$$\tilde{h}(t) = \tau t + 4\delta\tau e^{-t/2} - t - \frac{1}{2} \ln(1 + \delta\tau e^{-t/2}).$$

The same arguments imply that \tilde{h} satisfies the required conditions provided δ is small.

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